[…]

2. A Lanczos-type MIMO method obtained as Block-Arnoldi generalization.

Briefly:
Having as a starting point the MIMO descriptor system

$$\begin{cases}\mathbf{C}\dot{\mathbf{x}}(t) = -\mathbf{G}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) = \mathbf{L}^{\mathrm{T}}\mathbf{x}(t)\end{cases} \quad \text{respectively} \quad \begin{cases}s\mathbf{C}\mathbf{x}(s) = -\mathbf{G}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s)\\ \mathbf{y}(s) = \mathbf{L}^{\mathrm{T}}\mathbf{x}(s)\end{cases} \tag{6}$$

(where the dimension of the matrices are: $\mathbf{C}, \mathbf{G} \in \mathbf{R}^{NxN}$, $\mathbf{B} \in \mathbf{R}^{Nxm}$, $\mathbf{L} \in \mathbf{R}^{Nxm}$, **B** and **L** are full rank matrices), Lanczos type MIMO method sets up two biorthonormal bases $\mathbf{V}=[\mathbf{V}_1, \mathbf{V}_2,\dots, \mathbf{V}_k]$ **,** $\mathbf{W}=[\mathbf{W}_1,\mathbf{W}_2, ,\dots, \mathbf{W}_k]$ which determine in each iteration **k** a reduced **n**-dimensional (**n=km**) model of the initial system:

$$\begin{cases}\mathbf{C}_n\dot{\mathbf{x}}(t) = -\mathbf{G}_n\mathbf{x}(t) + \mathbf{B}_n\mathbf{u}(t)\\ \mathbf{y}(t) = \mathbf{L}_n^{\mathrm{T}}\mathbf{x}(t)\end{cases} \quad \text{respectively} \quad \begin{cases}s\mathbf{C}_n\mathbf{x}(s) = -\mathbf{G}_n\mathbf{x}(s) + \mathbf{B}_n\mathbf{u}(s)\\ \mathbf{y}(s) = \mathbf{L}_n^{\mathrm{T}}\mathbf{x}(s)\end{cases} \tag{7}$$

where $\mathbf{G}_n = \mathbf{W}^{\mathrm{T}}\mathbf{G}\mathbf{V},\ \mathbf{C}_n = \mathbf{W}^{\mathrm{T}}\mathbf{C}\mathbf{V},\ \mathbf{B}_n = \mathbf{W}^{\mathrm{T}}\mathbf{B},\ \mathbf{L}_n = \mathbf{V}^{T}\mathbf{L}$.

2.1. Markov moments expressions for MIMO systems.

Consider:
i) $s_0 \in \mathbf{C}$ such that the matrix **(G+s₀C)** is nonsingular.
ii) $\mathbf{F}_1, \mathbf{F}_2 \in \mathbf{C}^{NxN}$ two nonsingular matrices that form a factorization of the matrix $\mathbf{G}+\mathbf{s}_0\mathbf{C}$, i.e. $\mathbf{G} + \mathrm{s}_0\, \mathbf{C} = \mathbf{F}_1\mathbf{F}_2$.

iii) $\mathbf{H}(s) = \sum_{0}^{\infty}\mathbf{M}_{\mathrm{i}}(s - s_0)^{\mathrm{i}}$ the Taylor expansion of the transfer function of the initial system (7) around the complex number s_0.

Let us denote: $\mathbf{A} = -\mathbf{F}_1^{-1}\mathbf{C}\mathbf{F}_2^{-1};\ \mathbf{R} = \mathbf{F}_1^{-1}\mathbf{B};\ \mathbf{L} = \mathbf{F}_2^{-\mathrm{T}}\mathbf{E};\ s = \mathrm{s}_0 + \sigma$.

Hence $\mathbf{H}(s)$ can be rewritten as:

$$\mathbf{H}(s) = \mathbf{L}^{\mathrm{T}}(\mathbf{I}\text{-}(s - \mathrm{s}_0)\mathbf{A})^{-1}\mathbf{R} = \mathbf{L}^{\mathrm{T}}(\mathbf{I}\text{-}\sigma\mathbf{A})^{-1}\mathbf{R}$$

and consequently

$$\mathbf{H}(s)=\mathbf{L}^{\mathrm{T}}\left(\sum_{\mathrm{i}=0}^{\infty}A^{\mathrm{i}}(s-\mathrm{s}_0)^{\mathrm{i}}\right)\mathbf{R}=\sum_{\mathrm{i}=0}^{\infty}\mathbf{L}^{\mathrm{T}}\mathbf{A}^{\mathrm{i}}\mathbf{R}\sigma^{\mathrm{i}},$$

where the matriceal coeficients $\mathbf{M}_{\mathrm{i}} = \mathbf{L}^{\mathrm{T}}\mathbf{A}^{\mathrm{i}}\mathbf{R}$ are Markov moments. The relation above proves that describing the input-output behaviours of system (6) around

$s=s_0$ is equivalent with describing the input-output behavior of the system $\begin{cases}(\mathbf{I}-\sigma\mathbf{A})\mathbf{x}(\sigma)=\mathbf{R}\mathbf{u}(\sigma)\\ \mathbf{y}(\sigma)=\mathbf{L}^{\mathrm{T}}\mathbf{x}(\sigma)\end{cases}$ around s=0.

2.2. The algorithm.

Consider **A**, **R**, **L** as defined above with $\mathbf{F}_1\mathbf{F}_2$ being the trivial factorization of the nonsingular pencil $\mathbf{G}+s_0\mathbf{C}$, i.e. $\mathbf{F}_1\mathbf{F}_2 = (\mathbf{G}+s_0\mathbf{C})\mathbf{I}_N$ and $\mathbf{K}_k(\mathbf{A},\mathbf{R})=\mathbf{span}\{\mathbf{R}, \mathbf{AR}, \mathbf{A}^2\mathbf{R}, \ldots, \mathbf{A}^{k-1}\mathbf{R}\}$, $\mathbf{K}_k(\mathbf{A}^{\mathrm{T}},\mathbf{L})=\mathbf{span}\{\mathbf{L}, \mathbf{A}^{\mathrm{T}}\mathbf{L}, (\mathbf{A}^{\mathrm{T}})^2\mathbf{L}, \ldots, (\mathbf{A}^{\mathrm{T}})^{k-1}\mathbf{L}\}$ being k-order Krylov subspaces generated by **A** and **R**, respectively by $\mathbf{A}^{\mathrm{T}}$ and **L**.The following iterative algorithm sets up the biorthonormal bases **V** and **W** in the right and left Krylov subspaces $\mathbf{K}_k(\mathbf{A},\mathbf{R})$ respectively $\mathbf{K}_k(\mathbf{A}^{\mathrm{T}},\mathbf{L})$:

1. $\mathbf{V}_1^2=\mathbf{R}\ ;\ \mathbf{W}_1^2=\mathbf{L}\ ;\ \mathbf{V}_0=0; \mathbf{W}_0=0; \mathbf{D}_1=0;$
2. Compute QR[1] decomposition of $\mathbf{V}_1^2$ and $\mathbf{W}_1^2$:
   $\mathbf{V}_1^2=\mathbf{V}_1^1\mathbf{D}_1^1; \mathbf{W}_1^2=\mathbf{W}_1^1\mathbf{B}_1^1;$
3. Compute SVD[2] of $\left(\mathbf{W}_1^1\right)^{\mathrm{T}}\mathbf{V}_1^1=\mathbf{P}_1\mathbf{S}_1\mathbf{Q}_1^{\mathrm{T}};$
4. $\mathbf{V}_1=\mathbf{V}_1^1\mathbf{Q}_1\mathbf{S}_1^{-\frac{1}{2}}; \mathbf{W}_1=\mathbf{W}_1^1\mathbf{P}_1^{\mathrm{T}}\mathbf{S}_1^{-\frac{1}{2}}\ ;$
   $\mathbf{C}_1=\mathbf{W}_1^{\mathrm{T}}\mathbf{A}\mathbf{V}_1\ ;$
5. For $j=1,\ldots,k$ Do:
   6. $\mathbf{V}_{j+1}^2=\mathbf{A}\mathbf{V}_j-\mathbf{V}_j\mathbf{C}_j-\mathbf{V}_{j-1}\mathbf{B}_j;$
      $\mathbf{W}_{j+1}^2=\mathbf{A}^{\mathrm{T}}\mathbf{W}_j-\mathbf{W}_j\mathbf{C}_j^{\mathrm{T}}-\mathbf{W}_{j-1}\mathbf{D}_j^{\mathrm{T}};$
   7. Compute QR decomposition of $\mathbf{V}_{j+1}^2$ and $\mathbf{W}_{j+1}^2$:
      $\mathbf{V}_{j+1}^2=\mathbf{V}_{j+1}^1\mathbf{D}_{j+1}^1; \mathbf{W}_{j+1}^2=\mathbf{W}_{j+1}^1\left(\mathbf{B}_{j+1}^1\right)^{\mathrm{T}};$
   8. Compute SVD of $\left(\mathbf{W}_{j+1}^1\right)^{\mathrm{T}}\mathbf{V}_{j+1}^1=\mathbf{P}_{j+1}\mathbf{S}_{j+1}\mathbf{Q}_{j+1}^{\mathrm{T}};$

[1] QR Decomposition – B. Dumitrescu, C. Popeea, B. Jora; *Metode de calcul numeric matriceal. Algoritmi fundamentali,* All Educational, 1998, Bucharest, p. 150, Theorem 3.2.
[2] Singular Values Decomposition – B. Dumitrescu, C. Popeea, B. Jora; *Metode de calcul numeric matriceal. Algoritmi fundamentali,* All Educational, 1998, Bucharest, p.370, Theorem 5.1.

9. $\mathbf{B}_{j+1} = \mathbf{B}^1_{j+1}\mathbf{P}_{j+1}\mathbf{S}^{\frac{1}{2}}_{j+1}$; $\mathbf{D}_{j+1} = \mathbf{S}^{\frac{1}{2}}_{j+1}\mathbf{Q}^{\mathrm{T}}_{j+1}\mathbf{D}^1_{j+1}$;

10. $\mathbf{V}_{j+1} = \mathbf{V}^1_{j+1}\mathbf{Q}_{j+1}\mathbf{S}^{-\frac{1}{2}}_{j+1}$; $\mathbf{W}_{j+1} = \mathbf{W}^1_{j+1}\mathbf{P}_{j+1}\mathbf{S}^{-\frac{1}{2}}_{j+1}$;

11. End Do.

In the stated algoritm above only the upper indices on **S** are exponents but on the other symbols (**V**, **W**, **B**, **D**) they aren't exponents.

Two breakdown situations are posible in the Lanczos-type MIMO algorithm:

- first, in the stage 8, both $\mathbf{W}^1_{j+1}$ and $\mathbf{V}^1_{j+1}$ must be full-rank matrices. To detect and replace (almost) linearly dependent columns in these matrices, a deflation procedure must be implemented in the algorithm.
- secondly, in the same stage, the product $\left(\mathbf{W}^1_{j+1}\right)^{\mathrm{T}}\mathbf{V}^1_{j+1}$ must be a square full-rank matrix.

Until the breakdown occurs, the relations in stages 6,7,10 determine:

$$\begin{cases} \mathbf{AV} = \mathbf{VT}_k + \left[\mathbf{O}_{Nxm}\,\mathbf{O}_{Nxm}\ldots\mathbf{O}_{Nxm}\,\mathbf{V}_{k+1}\mathbf{D}_{k+1}\right]; \\ \mathbf{A}^{\mathrm{T}}\mathbf{W} = \mathbf{WT}_k + \left[\mathbf{O}_{Nxm}\,\mathbf{O}_{Nxm}\ldots\mathbf{O}_{Nxm}\,\mathbf{W}_{k+1}\mathbf{B}^{\mathrm{T}}_{k+1}\right]; \end{cases}$$

where **k** is the current iteration, **n=km** and the right and left Krylov subspaces bases $\mathbf{V} = [\mathbf{V}_1\,\mathbf{V}_2\,......\,\mathbf{V}_\mathrm{k}]$, and $\mathbf{W} = [\mathbf{W}_1\,\mathbf{W}_2\,......\,\mathbf{W}_\mathrm{k}]$ are biorthonormal $\left(\mathbf{W}^{\mathrm{T}}\mathbf{V} = \mathbf{I}_\mathrm{n}\right)$ and consequently

$$\mathbf{W}^{\mathrm{T}}\mathbf{AV} = \mathbf{T}_k \tag{8}$$

and the matrix $\mathbf{T}_k$ is block-tridiagonal:

$$\mathbf{T}_\mathrm{k} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{B}_2 & & & & \\ \mathbf{D}_2 & \mathbf{C}_2 & \mathbf{B}_3 & & & \\ & \mathbf{D}_3 & \mathbf{C}_3 & \mathbf{B}_4 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & \mathbf{D}_{k-1} & \mathbf{C}_{k-1} & \mathbf{B}_k \\ & & & & \mathbf{D}_k & \mathbf{C}_k \end{bmatrix}.$$

Proposition 3. The linear operator $\mathbf{P} = \mathbf{VW}^{\mathrm{T}}$; $\mathbf{P}:\mathbf{R}^{\mathrm{N}} \to \mathbf{K}_k(\mathbf{A},\mathbf{R})$ has the following properties:

i) $\mathbf{VW}^{\mathrm{T}}\mathbf{VW}^{\mathrm{T}} = \mathbf{VW}^{\mathrm{T}}$ i.e. $\mathbf{P}^2 = \mathbf{P}$,

ii) $\forall \mathbf{v} \in \mathbf{K}_k(\mathbf{A},\mathbf{r})$: $\mathbf{VW}^{\mathrm{T}}\mathbf{v} = \mathbf{v}$ i.e. $\mathbf{P}|_{\mathbf{K}_k(\mathbf{A},\mathbf{r})} = \mathbf{I}_N$, (9)

iii) $\forall \overline{\mathbf{w}} \in \mathbf{K}_k^{\perp}(\mathbf{A}^{\mathrm{T}},\mathbf{l})$: $\mathbf{VW}^{\mathrm{T}}\overline{\mathbf{w}} = 0$ i.e. $\mathbf{P}|_{\mathbf{K}_k^{\perp}(\mathbf{A}^{\mathrm{T}},\mathbf{l})} = \mathbf{O}_n$.

Therefore $\mathbf{VW}^{\mathrm{T}}$ is the projector from $\mathbf{R}^N$ on $\mathbf{K}_k(\mathbf{A},\mathbf{r})$ orthogonal to $\mathbf{K}_k(\mathbf{A}^{\mathrm{T}},\mathbf{l})$.

2.3. The reduced order model realization through the Lanczos-type MIMO algorithm.

The following will demonstrate that the relations (8),(9) in themselves determine an n-dimensional (n=km) reduced model of the initial system (6).

$$\mathbf{W}^{\mathrm{T}}\mathbf{AV} = \mathbf{T}_k \Rightarrow -\mathbf{W}^{\mathrm{T}}(\mathbf{G}+s_0\mathbf{C})^{-1}\mathbf{CV} = \mathbf{T}_k \Rightarrow$$

$$-\mathbf{W}^{\mathrm{T}}(\mathbf{G}+s_0\mathbf{C})^{-1}\mathbf{VW}^{\mathrm{T}}\mathbf{CV} = \mathbf{T}_k \Rightarrow -(\mathbf{W}^{\mathrm{T}}(\mathbf{G}+s_0\mathbf{C})\mathbf{V})^{-1}\mathbf{W}^{\mathrm{T}}\mathbf{CV} = \mathbf{T}_k \Rightarrow$$

$$-(\mathbf{W}^{\mathrm{T}}\mathbf{GV}+s_0\mathbf{W}^{\mathrm{T}}\mathbf{CV})^{-1}\mathbf{W}^{\mathrm{T}}\mathbf{CV} = \mathbf{T}_k \Rightarrow \mathbf{A}_n = -(\mathbf{G}_n + s_0\mathbf{C}_n)^{-1}\mathbf{C}_n = \mathbf{T}_k$$

where $\mathbf{G}_n = \mathbf{W}^{\mathrm{T}}\mathbf{GV}$ and $\mathbf{C}_n = \mathbf{W}^{\mathrm{T}}\mathbf{CV}$.

Moreover, $\forall\, j \in \overline{0,2n-1}$:

$$\mathbf{M}_j = \mathbf{L}^{\mathrm{T}}\mathbf{A}^j\mathbf{R} = \mathbf{L}^{\mathrm{T}}\mathbf{VW}^{\mathrm{T}}\mathbf{A}^j\mathbf{VW}^{\mathrm{T}}\mathbf{R} = \left(\mathbf{V}^{\mathrm{T}}\mathbf{L}\right)^{\mathrm{T}}\mathbf{T}_k^j\mathbf{W}^{\mathrm{T}}\mathbf{R} =$$

$$= \mathbf{L}_n^{\mathrm{T}}\mathbf{T}_k^j\mathbf{W}^{\mathrm{T}}(\mathbf{G}+s_0\mathbf{C})^{-1}\mathbf{VW}^{\mathrm{T}}\mathbf{B} = \mathbf{L}_n^{\mathrm{T}}\mathbf{T}_k^j\left(\mathbf{W}^{\mathrm{T}}(\mathbf{G}+s_0\mathbf{C})\mathbf{V}\right)^{-1}\left(\mathbf{W}^{\mathrm{T}}\mathbf{B}\right) =$$

$$= \mathbf{L}_n^{\mathrm{T}}\mathbf{T}_k^j\left(\mathbf{W}^{\mathrm{T}}\mathbf{GV}+s_0\mathbf{W}^{\mathrm{T}}\mathbf{CV}\right)^{-1}\left(\mathbf{W}^{\mathrm{T}}\mathbf{B}\right) = \mathbf{L}_n^{\mathrm{T}}\mathbf{T}_k^j(\mathbf{G}_n + s_0\mathbf{C}_n)^{-1}\mathbf{B}_n =$$

$$= \mathbf{L}_n^{\mathrm{T}}\mathbf{A}_n^j\mathbf{R}_n = \mathbf{M}_{n,j} ,$$

where $\mathbf{G}_n = \mathbf{W}^{\mathrm{T}}\mathbf{GV}$, $\mathbf{C}_n = \mathbf{W}^{\mathrm{T}}\mathbf{CV}$, $\mathbf{B}_n = \mathbf{W}^{\mathrm{T}}\mathbf{B}$, $\mathbf{L}_n = \mathbf{V}^T\mathbf{L}$.

Hence we proved the following theorem:

Theorem 2. (Padé via Lanczos-type MIMO algorithm) The setting up of the matrices **V**, **W**, by which the Lanczos-type MIMO iterative process applied to the initial system (6) results in a reduced **n**-dimensional (**n=km**) model (7) after **k** iterations , is equivalent with finding a local **2n**-order Padé approximation of the transfer function of the initial system, and also with an oblique projection of the initial system on $\mathbf{K}_n(\mathbf{A}, \mathbf{R})$ along the subspace $\mathbf{K}_n^{\perp}(\mathbf{A}^{\mathrm{T}},\mathbf{L})$. **[…]**